Boundary-Layer Tables for Similar Compressible Flow

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Abstract

THE similar solutions of the compressible, laminar boundary-layer equations depend on a pressure gradient parameter β and on the wall-to-inviscid stagnation temperature ratio g_w. However, derived quantities, such as various thicknesses, also depend on a speed parameter S. A comprehensive table of results is therefore three-dimensional and, in fact, does not exist. One of the more extensive tabulations is due to Back.[1] His three-dimensional table, however, is scanty, lacking results for negative β and for $g_w > 1$. Furthermore, the spacing on g_w and S is nonuniform and inadequate for interpolation or extrapolation. These remarks are not criticisms, since a comprehensive three-dimensional table is a prohibitive undertaking and, even it is existed, would be difficult and awkard to use.

We provide a new formulation that enables all quantities of interest to be determined by a set of two-dimensional tables in which β and g_w are the only entree parameters. With such a set, accurate values can be found for the skin-friction coefficient, Stanton number, and the five most common viscous and thermal boundary-layer thicknesses for arbitrary values of S. We use the compressible similarity equations under the traditional assumptions of steady flow of a perfect gas with unity values for the Prandtl number and Chapman-Rubesin parameter. The bounding wall is assumed impermeable, and the flow is two-dimensional or axisymmetric. Our nomenclature is usually that of Ref. 1 or may be found in the full paper.

Contents

We assume a numerical solution can be obtained for the transformed boundary-layer equations

$$f''' + ff'' + \beta[g_w + (1-g_w)G - f'^2] = 0 \tag{1a}$$

$$G'' + fG' = 0 \tag{1b}$$

subject to the boundary conditions

$$f(0) = f'(0) = G(0) = 0 \tag{2a}$$

$$f'(\infty) = G(\infty) = 1 \tag{2b}$$

The quantities of interest are the skin-friction coefficient, Stanton number, and several viscous and thermal thicknesses. For these quantities, we need $f''_w(\beta, g_w)$ and $G'_w(\beta, g_w)$ from the solution of Eqs. (1) and (2), where

$$\beta = \frac{2\xi}{u_e}\frac{du_e}{d\xi}\left(1 + \frac{\gamma-1}{2}M_e^2\right), \quad g_w = \frac{T_w}{T_{0e}} \tag{3a}$$

Most of the boundary-layer thicknesses also depend on the speed parameter

$$S = \frac{(\gamma-1)M_e^2/2}{1+(\gamma-1)M_e^2/2} \tag{3b}$$

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For the subsequent derivation, we need the following collection of integrals, all of which are exact:

$$\int_0^\eta d\eta = \eta, \quad \int_0^\eta f' d\eta = f, \quad \int_0^\eta f'' d\eta = f'$$

$$\int_0^\eta f''' d\eta = f'' - f''_w, \quad \int_0^\eta ff'' d\eta = ff' - \int_0^\eta f'^2 d\eta$$

$$\int_0^\eta Gf' d\eta = fG + G' - G'_w$$

$$\int_0^\eta f'^2 d\eta = \frac{1}{1+\beta}(f'' + ff' + \beta g_w \eta - f''_w) + \frac{\beta(1-g_w)}{1+\beta}\int_0^\eta G d\eta$$

$$\int_0^\eta f'^3 d\eta = \frac{2}{1+2\beta}\left[f'f'' + ½ ff'^2 + \beta f + \beta(1-g_w)(fG + G' - G'_w) - \int_0^\eta f''^2 d\eta\right] \tag{4}$$

These relations use Eqs. (1) and (2), and the last four also involve several integration by parts. In addition to Eqs. (3), two parameters are defined

$$C_v(\beta, g_w) = \int_0^\infty (1-f')d\eta = \lim_{\eta\to\infty}(\eta - f) \tag{5a}$$

$$C_t(\beta, g_w) = \int_0^\infty (1-G)d\eta = \lim_{\eta\to\infty}\left(\eta - \int_0^\eta G\, d\eta\right) \tag{5b}$$

where C_v is used in the incompressible theory. These parameters are evaluated at the time Eqs. (1) and (2) are solved.

In addition to arc length s and ξ, we require a scaled wall length $\bar{x}$ and a Reynolds number,

$$\bar{x} = \frac{\xi}{(\rho\mu u)_e r_w^{2\sigma}}, \qquad Re_{\bar{x}} = \left(\frac{\rho u}{\mu}\right)_e \bar{x}$$

It is also convenient to introduce the parameter

$$\Upsilon = \left(\frac{Re_{\bar{x}}}{2}\right)^{½}\frac{1}{\bar{x}} = \frac{(\rho u)_e r_w^\sigma}{(2\xi)^{½}}$$

The transverse distance n is then related to the similarity variable by means of

$$\Upsilon n = \int_0^\eta \frac{\rho_e}{\rho} d\eta = \int_0^\eta \frac{g_w + (1-g_w)G - Sf'^2}{1-S} d\eta$$

Five boundary-layer thicknesses are defined as follows:

δ = velocity thickness = n when $f' = 0.99$ and $\eta = \eta_{ev}(\beta, g_w)$

$$\Upsilon\delta = \int_0^{\eta_{ev}} \left[\frac{g_w + (1-g_w)G - Sf'^2}{1-S}\right] d\eta \tag{6}$$

δ_t = thermal thickness = n when $G = 0.99$ and $\eta = \eta_{et}(\beta, g_w)$

$$\Upsilon\delta_t = \int_0^{\eta_{et}} \left[\frac{g_w + (1-g_w)G - Sf'^2}{1-S}\right] d\eta \tag{7}$$

$$\delta^* = \text{displacement thickness} = \int_0^\infty \left[1 - \frac{\rho u}{(\rho u)_e}\right] dn$$

$$\Upsilon\delta^* = \int_0^\infty \left[\frac{g_w + (1-g_w)G - Sf'^2}{1-S} - f'\right] d\eta \tag{8}$$

$$\theta = \text{momentum defect thickness} = \int_0^\infty \frac{\rho u}{(\rho u)_e}\left(1-\frac{u}{u_e}\right)\mathrm{d}n$$

$$\Upsilon\theta = \int_0^\infty f'(1-f')\mathrm{d}\eta \tag{9}$$

ϕ = stagnation enthalpy defect thickness

$$= \int_0^\infty \frac{\rho u}{(\rho u)_e}\left(1-\frac{h_0-h_{0w}}{h_{0e}-h_{0w}}\right)\mathrm{d}n$$

$$\Upsilon\phi = \int_0^\infty f'(1-G)\mathrm{d}\eta \tag{10}$$

The velocity and thermal transformed edge values η_{ev} and η_{et} are functions only of β and g_w.

An overshoot[2] is experienced by f' when $\beta>0$ and $g_w>1$. When there is no overshoot or the overshoot is too small to be discernible, we use a conventional definition for η_{ev},

$$f'(\eta_{ev}) = 0.99, f'(\eta_m) \leq 1.000001$$

where η_m is the η value where f' is a maximum. For a discernible maximum, the smaller of the two η_{ev} given by

$$f'(\eta_{ev}) = 0.9+0.1f'(\eta_m), \quad \eta_{ev}>\eta_m$$

$$f'(\eta_{ev}) = 1.01, \quad \eta_{ev}>\eta_m$$

is used.

The infinite upper limit is replaced in Eqs. (8-10) with η and Eqs. (4) are used to evaluate the resulting integrals. Whenever a G integral or an f appears, these are replaced by Eqs. (5). After simplification, the $\eta\to\infty$ limit then is taken. This derivation then yields the following exact results:

$$\Upsilon\delta^* = \frac{1}{(1+\beta)(1-S)}\{Sf''_w + [1+(1-S)\beta] \times [C_v-(1-g_w)C_t]\} \tag{11}$$

$$\Upsilon\theta = \frac{1}{1+\beta}\{f''_w - \beta[C_v-(1-g_w)C_t]\} \tag{12}$$

$$\Upsilon\phi = G'_w \tag{13}$$

To establish analogous relations for δ and δ_t, the right sides of Eqs. (6) and (7) are written as

$$I = \frac{1}{1-S}\int_0^{\eta^*} [g_w+(1-g_w)G-Sf'^2]\mathrm{d}\eta$$

where η^* is either η_{ev} or η_{et}. We assume η^* is sufficiently large so that Eqs. (5), $f'(\eta^*)=1$, and $f''(\eta^*)=0$ approximately hold. The accuracy of these approximations, which are used only for δ and δ_t, is shown in the full paper to be excellent. With these relations and Eqs. (4), we obtain

$$I = \eta^* + \frac{1}{(1+\beta)(1-S)}\{S(f''_w+C_v) - (1-g_w)[1+\beta(1-S)]C_t\}$$

Consequently, δ and δ_t are given by

$$\Upsilon\delta = \eta_{ev} + \frac{1}{(1+\beta)(1-S)}\{S(f''_w+C_v) - (1-g_w)[1+\beta(1-S)]C_t\} \tag{14}$$

$$\Upsilon\delta_t = \eta_{et} + \frac{1}{(1+\beta)(1-S)}\{S(f''_w+C_v) - (1-g_w)[1+\beta(1-S)C_t]\} \tag{15}$$

Equations (11-15) are the desired result. By tabulating η_{ev}, η_{et}, C_v, C_t, f''_w, and G'_w in terms of β and g_w, the foregoing thicknesses can be found for an arbitrary value of S. For completeness, we also provide the wall heat transfer, Stanton number, wall shear stress, and skin-friction coefficient as

$$q_w = k_w\left(\frac{\partial T}{\partial n}\right)_w = \frac{h_{0e}-h_{0w}}{(2Re_{\bar{x}})^{1/2}}(\rho u)_e G'_w$$

$$St = \frac{q_w}{(h_{0e}-h_{0w})(\rho u)_e} = \frac{G'_w}{(2Re_{\bar{x}})^{1/2}}$$

$$\tau_w = \mu_w\left(\frac{\partial u}{\partial n}\right)_w = \frac{(\rho u^2)_e}{(2Re_{\bar{x}})^{1/2}}f''_w$$

$$c_f = \frac{2\tau_w}{(\rho u^2)_e} = \left(\frac{2}{Re_{\bar{x}}}\right)^{1/2} f''_w$$

When the wall is adiabatic, G is undefined, although $g(\eta)=1$ and h_0 is a constant across the boundary layer. As a consequence, C_t and G'_w are discarded. However, the results for f''_w, c_f, C_v, and η_{ev} are unaltered and thus hold for an adiabatic wall. An adiabatic wall temperature profile

$$\frac{T}{T_e} = \frac{1-Sf'^2}{1-S} \tag{16}$$

is used in conjunction with a temperature thickness $\tilde{T}_{et}$, defined by

$$\frac{T_{0e}-\tilde{T}_{et}}{T_{0e}-T_e} = 0.9801 \tag{17}$$

where T_{0e} is also the temperature of the gas that is adjacent to the wall and a tilde denotes an adiabatic wall. Equations (16) and (17) result in $f'(\tilde{\eta}_{et})=0.99$. Hence, $\tilde{\eta}_{et}$ is given by $\tilde{\eta}_{et}=\eta_{ev}(\beta,1)$. We chose the 0.9801 value in Eq. (17) so that additional tables for an adiabatic wall are unnecessary. From Eq. (7), we have

$$\Upsilon\tilde{\delta}_t = \int_0^{\tilde{\eta}_{et}} \frac{1-Sf'^2}{1-S}\mathrm{d}\eta$$

which becomes

$$\Upsilon\tilde{\delta}_t = \eta_{ev}(\beta,1) + \frac{S}{(1+\beta)(1-S)}(f''_w+C_v)$$

Aside from the replacement of $\tilde{\eta}_{et}$ with $\eta_{ev}(\beta,1)$, this agrees with Eq. (15) with $g_w=1$. Since the rightmost integral in the last of Eqs. (4) has not been evaluated, the thickness

$$\Upsilon\tilde{\phi} = \int_0^\infty f'(1-f'^2)\mathrm{d}\eta$$

is not used. Thus, only $\tilde{\eta}_{et}$ and $\tilde{\delta}_t$ are appropriate for an adiabatic wall.

The full paper (obtainable from the authors) contains a more detailed derivation of the equations as well as a discussion of the numerical method. In addition, a comprehensive set of tables is included, starting with the separation value $\beta_{sp}(g_w)$ for $0\leq g_w\leq 5$. The next six tables provide C_v, C_t, η_{ev}, η_{et}, f''_w, and G'_w for the range

$$\beta_{sp}\leq\beta\leq 100, \quad 0\leq g_w\leq 5$$

with closely spaced β and g_w values. Since the dependence on S is analytical, these boundary-layer tables represent the most comprehensive set produced to date.

References

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[2]Mcleod, J. B. and Serrin, J., "The Behavior of Similar Solutions in a Compressible Boundary Layer," *Journal of Fluid Mechanics*, Vol. 34, Nov. 1968, pp. 337-342.